First-order correction to the Casimir force within an inhomogeneous medium

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For the Casimir piston filled with an inhomogeneous medium, we regularized and expressed the Casimir energy with cylinder kernel coefficients by using the first-order perturbation theory. When the refractive index of the medium is smoothly inhomogeneous (i.e., derivatives of all orders exist), a logarithmically cutoff-dependent term and a quadratically cutoff-dependent term in the Casimir energy are found. We show that in the piston model these terms vanish in the force and thus the Casimir force is always cutoff independent, but these terms will remain in the force in the half-space model and must be removed by additional regularizations. We give explicit benchmark solutions to the first-order corrections of both Casimir energy and Casimir force for an exponentially decaying profile. The present method can be extended to other inhomogeneous profiles. Our results should be useful for future relevant calculations and experimental studies.

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I. INTRODUCTION

The Casimir effect [1,2] is known as one of the direct manifestations of vacuum zero-point energy in quantum physics. A mode-summation method can be used to predict easily an attractive force between two electrically neutral, perfectly conductive half-spaces. Following Casimir's pioneering work, many other approaches such as Lifshitz formulation [3–6], source theory [7], scattering theory [8–13], and so on have been developed to extend the study of the Casimir effect to real materials [14], finite temperatures [15], and curved geometries [16–19]. However, the Casimir force in a variety of scenarios is involved with many divergence problems during calculations, in contrast with Casimir's work, due to the geometry of boundaries or topology of space [20–22]. These unresolved divergences usually show a logarithmic cutoff dependence and seem to be irremovable (while other quartic or cubic cutoff-dependent terms in Casimir energy are already well understood as volume energy or surface energy, and thus are removable) [23,24].

When the inhomogeneity of the medium rather than the complicated geometry of objects in the Casimir apparatus is considered, it is known that analytical description of the Casimir force (stress or force density) has already been obtained [25,26], without an explicit divergence problem. The results in both references have subtracted the "bulk contribution" and thus relate only to the scattering Green's function. This, in our minds, regularizes the quartic diverging term (volume energy) in the Casimir energy, but might not be a thorough regularization, as also mentioned in Refs. [25,27]. In fact, divergences indeed occur when one tries to evaluate the numeric values of both the Casimir energy and the Casimir force, according to the obtained analytical description within an inhomogeneous medium [25]. Efforts have been made, since then, to understand the remaining divergence [28], but the problem is still far from solved.

We note, in the model of half-spaces which is used in Ref. [25], cutoff-dependent terms in the Casimir energy must be assigned physical meanings and thus removed manually by introducing corresponding regularizations. While in the model of the Casimir piston, cutoff-dependent terms may vanish automatically in the Casimir force, due to the cancellation of contributions from the left and right cavities. We also note a naive mode-summation approach [29] in the first-order perturbation theory (which is supposed to show the cutoff dependence) turns out to yield cutoff-independent results of the Casimir force for inhomogeneous media. All of the above have led us to expect that more useful information could be obtained from the Casimir piston model. Therefore, we adopt the piston model here, following the mode-summation approach in the first-order perturbation theory as in Ref. [29] (but in a more general form), to investigate the Casimir physics within inhomogeneous media. The purposes for doing so are threefold. First, we want to analyze the inhomogeneity-induced Casimir divergence with the heat kernel expansion and expect to obtain some insights for additional regularizations for the half-space model. Second, we want to know if the cutoff independence of the Casimir force is true for various inhomogeneous profiles, instead of a particular case. Third, we want to show how large the influence of the inhomogeneity on the total Casimir force is. As weak force measurements have been developed [14,30,31] and reached a quite precise level (within 1%) [32], and the Casimir force between bodies in a liquid has also been measured [33,34], inhomogeneity-induced corrections may be useful for future experimental studies as the experimental configuration becomes more and more complicated.

In the present paper, we first derive general expressions of Casimir energy for inhomogeneous media in Sec. II. The summations in expressions of Casimir energy are organized and re-expanded over the cutoff parameter in Sec. III. Then we prove the cutoff independence of the Casimir force

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for smooth inhomogeneity in Sec. IV, where we also see logarithmic divergence in the half-space model. We investigate the exponentially decaying inhomogeneity in Sec. V and give first-order corrections to the Casimir energy and the Casimir force. Discussions and conclusions are given in Secs. VI and VII, respectively.

II. CASIMIR ENERGY OF PLATES WITHIN INHOMOGENEOUS MEDIA

In the mode-summation technique, the total zero-point energy of the Casimir piston device (see Fig. 1) is expressed as [29]

$$E_0 = \frac{1}{2}\sum_{m,p,q,\lambda}\omega_{m,p,p,\lambda} + \mathcal{L}\to\mathcal{R}, \tag{1}$$

where m, p, and q are indexes for three wave numbers (m for the direction perpendicular to the plates), λ is the index for polarization, the notation $\mathcal{L}\to\mathcal{R}$ represents the counterpart for the right cavity, and also we have set $\hbar = c = 1$, which will be recovered later according to dimensional analysis. For simplicity, we use $k_\parallel$ to account for $\{p,q\}$, t for $\{m,k_\parallel\}$, and J for $\{t,\lambda\}$. Below we focus on the left cavity and omit the notation $\mathcal{L}\to\mathcal{R}$ in all equations, just keeping in mind that the right counterpart should be added in the final step. We adopt the cutoff regularization, and Eq. (1) then becomes

$$\tilde{E} = \lim_{\xi\to 0}\frac{1}{2}\sum_J \omega_J e^{-\xi\omega_J}. \tag{2}$$

We omit the limit notation in the following for simplicity as well.

When there is an inhomogeneous perturbation in the refractive index $n(x) = n_0[1+\delta\alpha f(x)]$, where $\delta\alpha$ is a small perturbation value, the difference of the regularized Casimir energy is then

$$\delta\tilde{E} = \frac{1}{2}\partial_\xi \sum_J \xi\omega_J^1 e^{-\xi\omega_J^0}, \tag{3}$$

with $\omega_J^1 = -\delta\alpha\mathcal{P}_J\omega_J^0$ and $\mathcal{P}_J \equiv \langle\chi_J^0|f(x)|\chi_J^0\rangle$. χ_J^0 is the Jth unperturbed eigen wave function with ω_J^0 being its eigenfrequency. We can find the expressions for the electric fields χ_J^0 in Ref. [29] [Eqs. (14) and (15) after correction in their erratum; we do not use E for electric field here to avoid confusion with Casimir energy]. We have also considered the perturbation of frequency in the exponent, and thus we get a factor of $\partial_\xi\xi$ in our Eq. (3). However, it does not otherwise change our argument. Therefore, we can obtain

$$\delta\tilde{E} = -\frac{\delta\alpha}{2n_0}\partial_\xi\xi\sum_J \mathcal{P}_J k_J^0 e^{-\xi k_J^0/n_0}. \tag{4}$$

Here $k_J^0 = \sqrt{k_\parallel^2 + (m\pi/a)^2}$ is the wave number for homogeneous media defined as $k_J^0 \equiv n_0\omega_J^0$. The perturbation theory is justified as long as $\mathcal{P}_J$ is bounded and $\delta\alpha\mathcal{P}_J \ll 1$. We note that only $\mathcal{P}_J$ varies for different polarizations. To sum up polarizations first, we obtain

$$\sum_{\lambda=1,2}\mathcal{P}_J = (2-\delta_{m0})F_0 - (2-\delta_{m0})\frac{(m\pi/a)^2}{k_\parallel^2 + (m\pi/a)^2}F_m, \tag{5}$$

with the Fourier coefficient of the perturbation profile $f(x)$,

$$F_m \equiv \frac{1}{a}\int_0^a f(x)\cos\frac{2m\pi x}{a}dx, \tag{6}$$

where $\lambda = 1,2$ represents two different polarizations. To obtain the counterparts of our Eqs. (5) and (6) for the right cavity we can substitute a with $D-a$ and also change the integration range in Eq. (6) from $(0,a)$ to (a,D).

Substituting Eq. (5) into Eq. (4), we can split the Casimir energy into two parts for the convenience of calculation. The first part is

$$\begin{aligned}\delta\tilde{E}_1 &= -\frac{\delta\alpha}{2n_0}\partial_\xi\xi\sum_t(2-\delta_{m0})F_0k_t^0e^{-\xi k_t^0/n_0}\\ &= \frac{\delta\alpha A n_0^2}{4\pi}F_0\partial_\xi\xi\partial_\xi\hat{\Xi}(\xi)\sum_m(2-\delta_{m0})e^{-m\pi\xi/an_0},\end{aligned} \tag{7}$$

and the second part is

$$\begin{aligned}\delta\tilde{E}_2 &= \frac{\delta\alpha}{2n_0}\partial_\xi\xi\sum_t(2-\delta_{m0})\frac{(m\pi/a)^2}{k_\parallel^2+(m\pi/a)^2}F_mk_t^0e^{-\xi k_t^0/n_0}\\ &= \frac{\delta\alpha A n_0^2}{4\pi}\partial_\xi^3\sum_m(2-\delta_{m0})F_me^{-m\pi\xi/an_0},\end{aligned} \tag{8}$$

where $\hat{\Xi} \equiv \xi^{-2}(1-\xi\partial_\xi)$, $m = 0,1,2,\ldots$, and A is the surface area of plates. To get the second equalities for both Eqs. (7) and (8) we have integrated over $k_\parallel$, provided that the lateral dimension of the plates is much larger than the distance between plates, i.e., $\sqrt{A} \gg D$, so we can replace summation with the corresponding integral. The counterparts of Eqs. (7) and (8) for the right cavity can be obtained by substituting a with $D-a$.

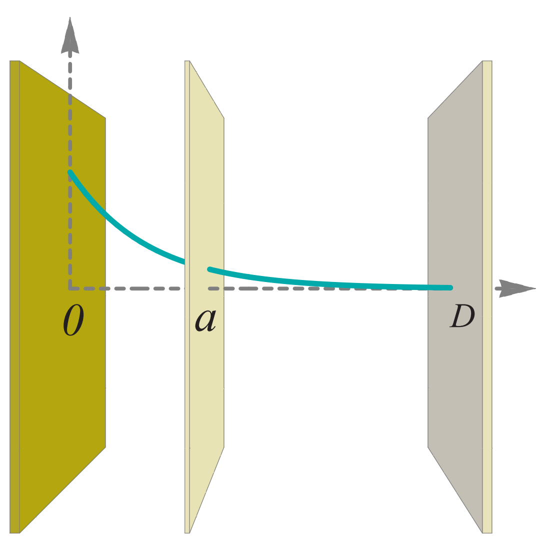

FIG. 1. (Color online) The Casimir piston model [29] with four (top, bottom, front, and back) hidden plates. The cyan curve represents an arbitrary inhomogeneous profile of the refractive index. All plates are of perfect conductivity.

On the other hand, following Eq. (2) and integrating over $k_\parallel$, we can also obtain the regularized Casimir energy for the homogeneous case:

$$\tilde{E}_h = -\frac{An_0^2}{4\pi}\partial_\xi\,\hat{\Xi}(\xi)\sum_m(2-\delta_{m0})e^{-m\pi\xi/an_0}. \tag{9}$$

Now, we can multiply Eqs. (7)–(9) with a factor of $\hbar c$, and add to them with the counterparts of the right cavity, and then take the $\xi\to 0$ limit. The sum of these three equations gives the total Casimir energy for the inhomogeneous case. Its derivative with respect to position a yields the Casimir force on the central plate. These three expressions are quite general for perturbation profiles as long as $\mathcal{P}_J$ is bounded and $\delta\alpha\mathcal{P}_J \ll 1$.

III. SUMMATION AND RE-EXPANSION

The summations in Eqs. (7) and (9) can be expressed in terms of a polylogarithm function:

$$\sum_{m=0}(2-\delta_{m0})e^{-m\pi\xi/an_0} = \left(\sum_{m=1}2e^{-m\pi\xi/an_0}\right)+(2-\delta_{00})$$
$$= 2\mathrm{Li}_0(e^{-\pi\xi/an_0})+1. \tag{10}$$

The summation in Eq. (8) depends on the particular form of the perturbation profile. However the $m=0$ term vanishes obviously. Then we can rewrite it as

$$\delta\tilde{E}_2 = \frac{\delta\alpha An_0^2}{2\pi}\partial_\xi^3\sum_{m=0\,\mathrm{or}\,1}F_m e^{-m\pi\xi/an_0}, \tag{11}$$

where m can run from 0 or 1 as needed. For the Taylor basis of order d,

$$f_d(x) = \left(\frac{x}{D}\right)^d, \tag{12}$$

where $d=0,1,2,\ldots$, according to the definition of F_m (we now consider m running from 1), we can integrate by parts to get the following relation for $d\geqslant 2$ for the left cavity:

$$F^{\mathcal{L}}_{m,d} = \frac{ad}{D(2m\pi)^2}\left(\frac{a}{D}\right)^{d-1}-\left(\frac{a}{2m\pi}\right)^2\frac{d(d-1)}{D^2}F^{\mathcal{L}}_{m,d-2}, \tag{13}$$

while we have $F^{\mathcal{L}}_{m,0}=F^{\mathcal{L}}_{m,1}=0$. The explicit expression is then

$$F^{\mathcal{L}}_{m,d} = \sum_{i=1}^{[d/2]}\frac{(-1)^{i-1}d!}{(d+1-2i)!}(2m\pi)^{-2i}\left(\frac{a}{D}\right)^{2i-1}\left(\frac{a}{D}\right)^{d+1-2i}, \tag{14}$$

where $[d/2]$ denotes the bare integer part of $d/2$. The counterpart for the right cavity can be obtained similarly:

$$F^{\mathcal{R}}_{m,d} = \sum_{i=1}^{[d/2]}\frac{(-1)^{i-1}d!}{(d+1-2i)!}(2m\pi)^{-2i}$$
$$\times\left(\frac{D-a}{D}\right)^{2i-1}\left[1-\left(\frac{a}{D}\right)^{d+1-2i}\right]. \tag{15}$$

We note that they all have the form $\sum_{i=1}^{[d/2]}g_i m^{-2i}$, where g_i's are some coefficients independent of m. Back to Eq. (11), we now know the summation is a combination of polylogarithm functions:

$$\delta\tilde{E}_2 = \frac{\delta\alpha An_0^2}{2\pi}\partial_\xi^3\sum_{i=1}^{[d/2]}g_i\mathrm{Li}_{2i}(e^{-\pi\xi/an_0}). \tag{16}$$

In general, these polylogarithm functions can be expanded over ξ (with $\xi\to 0$) as

$$\mathrm{Li}_0\left(e^{-\frac{\pi\xi}{an_0}}\right) = \frac{an_0}{\pi\xi}+\sum_{k=0}^{\infty}\frac{\zeta(-k)}{k!}\left(-\frac{\pi\xi}{an_0}\right)^k,$$
$$\mathrm{Li}_s\left(e^{-\frac{\pi\xi}{an_0}}\right) = \frac{1}{(s-1)!}\left(-\frac{\pi\xi}{an_0}\right)^{s-1}\left[H_{s-1}-\ln\left(\frac{\pi\xi}{an_0}\right)\right]$$
$$+\sum_{k=0,k\neq s-1}^{\infty}\frac{\zeta(s-k)}{k!}\left(-\frac{\pi\xi}{an_0}\right)^k, \tag{17}$$

where $s=2,4,6,\ldots$, and $H_s=\sum_{h=1}^{s}1/h$ is the harmonic number with $H_0=0$.

Before going further to get the explicit expression of the Casimir energy for the inhomogeneous case, we inspect another set of profiles,

$$f(x) = e^{-\eta x+\Delta}, \tag{18}$$

where $\mathrm{Re}[\eta]\geqslant 0$. The exponential profile is mathematically convenient and allows for direct comparison to previous studies [25] and if we set η to be purely imaginary it can also describe the sinusoidal profile which has also been studied previously [29]. For the left cavity we have (we now consider m running from 0)

$$F^{\mathcal{L}}_{m,\eta} = \frac{e^\Delta}{4\pi i}[1-e^{-\eta a}]\left[\frac{1}{m-i\eta a/2\pi}-\frac{1}{m+i\eta a/2\pi}\right]. \tag{19}$$

The right counterpart could be obtained by the substitution $(1-e^{-\eta a})\to(e^{-\eta a}-e^{-\eta D})$, $a\to(D-a)$. Equation (11) then becomes

$$\delta\tilde{E}_2 = \frac{\delta\alpha An_0^2}{2\pi}\frac{e^\Delta}{4\pi i}[1-e^{-\eta a}]$$
$$\times\partial_\xi^3\left[\phi\left(e^{-\frac{\pi\xi}{an_0}},1,-\frac{i\eta a}{2\pi}\right)-\phi\left(e^{-\frac{\pi\xi}{an_0}},1,\frac{i\eta a}{2\pi}\right)\right]. \tag{20}$$

Here the Lerch zeta function ϕ can be expanded over ξ (with $\xi\to 0$) as

$$\phi\left(e^{-\frac{\pi\xi}{an_0}},1,\beta\right) = \left[\sum_{k=0}\frac{\left(\frac{\pi\xi}{an_0}\beta\right)^k}{k!}\right]\left\{\sum_{k=1}\zeta(1-k,\beta)\frac{\left(-\frac{\pi\xi}{an_0}\right)^k}{k!}\right.$$
$$\left.+\left[\psi(1)-\psi(\beta)-\ln\left(\frac{\pi}{an_0}\right)-\ln\xi\right]\right\}, \tag{21}$$

where ψ is the digamma function and ζ is the Hurwitz zeta function. We note β is not any negative integer and thus $\zeta(1-k,\beta)$ and $\psi(\beta)$ are always finite. Furthermore, since only β is complex, we have $\phi(e^{-\pi\xi/an_0},1,\beta^*)=\phi^*(e^{-\pi\xi/an_0},1,\beta)$ and that is why Eq. (20) is always real.

IV. CUTOFF INDEPENDENCE

Up to now, we see the total Casimir energy for the inhomogeneous case is generally expressed as a Laurent-type series:

$$\tilde{E} = \tilde{E}_h + \delta\tilde{E}_1 + \delta\tilde{E}_2 = \sum_{i=-4} C_i \xi^i + \sum_{i=0} N_i \xi^i \ln \xi. \quad (22)$$

These coefficients C_i and N_i are well-studied heat kernel (or more precisely cylinder kernel here) coefficients in cutoff regularization [23,24,35]. They depend only on the geometry property and the boundary condition of the system under consideration. The divergent terms under limit $\xi \to 0$ are usually assigned to the self-energy of volume or surface and so on to renormalize the theory. Here, in our case, we should check these divergent terms, making sure that their coefficients are independent of position a so that these divergences do not go into the Casimir force, as we expect the observable—the Casimir force—to be finite. The constant term C_0 is the free Casimir energy, and its derivative with respect to position a is the Casimir force we want to calculate. Once these divergent terms are assured to be independent of position a, the final result of the Casimir force should be identical to the result from Ref. [36].

We now prove the cutoff independence of the Casimir force for any smoothly inhomogeneous perturbation profile. We know any smooth function in domain $(0,D)$ can be expanded via the Taylor bases given in Eq. (12). Therefore, equivalently what we need is to prove the cutoff independence of the Casimir force for the basis of any order d. We proceed in the following way.

First, the cutoff property of Eq. (7) is determined by $F_0 \partial_\xi \hat{\Xi}(\xi)(2\text{Li}_0 + 1)$ and the cutoff property of Eq. (9) is determined by $\partial_\xi \hat{\Xi}(\xi)(2\text{Li}_0 + 1)$ according to Eq. (10). We note the operators acting on the polylogarithm function generally decrease the power of ξ by 3 orders, and $\text{Li}_0 = \sum_{i=-1} l_i \xi^i$. Thus we have to check $i = -1$ to 3 to see the cutoff property. According to the expansion of the polylogarithm function, Eq. (17), we have

$$l_{-1}^{\mathcal{L}} \xi^{-1} = \frac{a n_0}{\pi \xi}, \quad l_{-1}^{\mathcal{R}} \xi^{-1} = \frac{(D-a) n_0}{\pi \xi}.$$

Thus for Eq. (9) we have

$$C_{-4}\xi^{-4} \propto A D n_0^3 \xi^{-4},$$

and for Eq. (7) we have

$$C_{-4}\xi^{-4} \propto A D n_0^3 \xi^{-4} \delta\alpha \frac{1}{D} \int_0^D f dx.$$

These two quartic divergent terms serve as the self-energy of the intermedia which is exactly of volume AD, and their independence of position a indicates they will not come into the Casimir force. Next, we also have

$$l_0^{\mathcal{L}} = l_0^{\mathcal{R}} = -\tfrac{1}{2},$$

which means $C_{-3}\xi^{-3} = 0$ for both Eqs. (7) and (9). This term is usually proportional to the surface area A and serves as the surface energy. The absence of the surface divergence term is due to the cancellation of TE and TM contributions as also reported in Ref. [22]. Next, we note, whatever $l_1 \xi^1$ is, we have $\hat{\Xi}(\xi) \cdot l_1 \xi^1 = 0$ and whatever $l_2 \xi^2$ is, we have $\partial_\xi \hat{\Xi}(\xi) \cdot l_2 \xi^2 = 0$. Thus $C_{-2}\xi^{-2} = C_{-1}\xi^{-1} = 0$. Therefore, the cutoff-dependent terms in Eqs. (7) and (9) all vanish in the Casimir force.

Now we turn to the contribution from Eq. (8), which has been expressed as Eq. (16). Since ∂_ξ^3 has a good property—all polynomial terms under it vanish when $\xi \to 0$, for positive plural $s = 2i$, we only need to consider the logarithm terms in Li_s to check the cutoff property. We have

$$\begin{aligned} & g_i^{\mathcal{L}} \left(\frac{\xi}{a}\right)^{2i-1} \ln \xi + g_i^{\mathcal{R}} \left(\frac{\xi}{D-a}\right)^{2i-1} \ln \xi \\ & \propto \ln \xi \left\{ \left(\frac{\xi}{a}\right)^{2i-1} \left(\frac{a}{D}\right)^{2i-1} \left(\frac{a}{D}\right)^{d+1-2i} \right. \\ & \quad \left. + \left(\frac{\xi}{D-a}\right)^{2i-1} \left(\frac{D-a}{D}\right)^{2i-1} \left[1 - \left(\frac{a}{D}\right)^{d+1-2i}\right] \right\} \\ & = \left(\frac{\xi}{D}\right)^{2i-1} \ln \xi. \end{aligned} \quad (23)$$

After performing ∂_ξ^3, we see term $i = 1$ contributes to $C_{-2}\xi^{-2}$ and term $i = 2$ contributes to $N_0 \ln \xi$. All other logarithm terms under ∂_ξ^3 vanish when $\xi \to 0$. We see $C_{-2} \propto \frac{1}{D}$ and $N_0 \propto \frac{1}{D^3}$, and thus both of them are independent of position a. However, in the half-space model there is no right cavity; therefore, those logarithm terms will appear to be a dependent and must be removed manually. Unfortunately, this is an unresolved problem yet, as we know.

This completes our proof that for the Taylor basis of any order d, the Casimir force for plates within an inhomogeneous medium is cutoff independent. Therefore, for any smooth inhomogeneity, which is a superposition of Taylor bases, the Casimir force will have the inherited cutoff independence.

V. APPLICATIONS

The Taylor expansion of the inhomogeneity profile is useful for cutoff analysis, but will result in a series of constant terms, C_0^d. This is inconvenient for calculation of the free Casimir energy and the Casimir force. Fortunately, for some cases, we are able to do the calculation without the Taylor expansion. One example is the profile given in Eq. (18) with $\eta > 0$ and $\Delta = 0$. Such a profile is common for fluids and gases in the gravitational field and is potentially experimentally achievable by engineering the density of a medium with acousto-optical techniques or other external fields.

The cutoff independence of $\tilde{E}_h$ and $\delta\tilde{E}_1$ can be analyzed exactly in the same way as above, while the $\delta\tilde{E}_2$ now is described by Eq. (20). Similarly, the logarithm terms

$$\begin{aligned} & (1 - e^{-\eta a}) \left(\frac{\beta^{\mathcal{L}} \xi}{a}\right)^k \ln \xi + (e^{-\eta a} - e^{-\eta D}) \left(\frac{\beta^{\mathcal{R}} \xi}{D-a}\right)^k \ln \xi \\ & \propto (1 - e^{-\eta D}) \xi^k \ln \xi, \end{aligned}$$

are independent of position a. Therefore, we have again the cutoff independence of the Casimir force, as expected.

To obtain the explicit expression of the Casimir force, we only need to calculate the C_0 term, which comes from ξ^3 in Li or ϕ. Now we have

$$F_0^{\mathcal{L}} = \frac{1}{\eta a}(1 - e^{-\eta a}), \quad F_0^{\mathcal{R}} = \frac{1}{\eta(D-a)}(e^{-\eta a} - e^{-\eta D}). \tag{24}$$

The C_0 contribution from $\tilde{E}_h$ is

$$-\frac{An_0^2}{4\pi}(-2)\left[\frac{2\zeta(-3)}{3!}\left(-\frac{\pi}{an_0}\right)^3 + \frac{2\zeta(-3)}{3!}\left(-\frac{\pi}{(D-a)n_0}\right)^3\right] = -\frac{A\hbar c\pi^2}{720n_0}\left[\frac{1}{a^3} + \frac{1}{(D-a)^3}\right]. \tag{25}$$

Here we have recovered $\hbar c$ in the end. The C_0 contribution from $\delta\tilde{E}_1$ is

$$\delta\alpha\frac{A\hbar c\pi^2}{720n_0}\left[F_0^{\mathcal{L}}\left(\frac{1}{a}\right)^3 + F_0^{\mathcal{R}}\left(\frac{1}{D-a}\right)^3\right]. \tag{26}$$

During the calculation of this term, we have seen that the operator $\partial_\xi\xi$ introduced by including the exponent in the regularization in Eq. (3) does not change the observable value (compared with the one without including the exponent in the regularization), as expected. The C_0 contribution from $\delta\tilde{E}_2$ is

$$\delta\alpha\frac{A\hbar c\pi^2}{720n_0}\left(\frac{1}{a}\right)^3\frac{180}{\pi}(1 - e^{-\eta a})\cdot \times \mathrm{Im}\left[\beta^3\left(\psi(1) - \ln\frac{\eta}{2n_0} + \frac{11}{6}\right) + \beta^3[\ln i\beta - \psi(\beta)] - \beta/12\right] + \mathcal{L} \to \mathcal{R}, \tag{27}$$

where $\beta = -\frac{i\eta a}{2\pi}$ and $\mathcal{L} \to \mathcal{R}$ represents the right counterpart where we should make the replacement $(1 - e^{-\eta a}) \to (e^{-\eta a} - e^{-\eta D})$ and the replacement $a \to (D - a)$ for the other a's. Im is the symbol for the imaginary part of an expression.

If we make the transforms $\eta \to -ib\frac{\pi}{D}$ and $\Delta \to i\Delta$ and make use of the real part of Eqs. (19) and (20), we can have some insights for profiles $f(x) = \cos(\frac{b\pi}{D}x + \Delta)$, where $0 \leqslant \Delta < 2\pi$ and $b > 0$. We have

$$F_{m,b}^{\mathcal{L}} = \frac{1}{4\pi}\left[\sin\Delta - \sin\left(2\pi\frac{ba}{2D} + \Delta\right)\right] \times \left[\frac{1}{m - ba/2\pi} - \frac{1}{m + ba/2\pi}\right]$$

and

$$\delta\tilde{E}_2 = \frac{\delta\alpha An_0^2}{2\pi}\frac{1}{4\pi}\left[\sin\Delta - \sin\left(2\pi\frac{ba}{2D} + \Delta\right)\right] \times \partial_\xi^3\left[\phi\left(e^{-\frac{\pi\xi}{an_0}}, 1, -\frac{ba}{2\pi}\right) - \phi\left(e^{-\frac{\pi\xi}{an_0}}, 1, \frac{ba}{2\pi}\right)\right].$$

Here we choose $b = 2$ and $\Delta = 0$. For $0 < a < D$ we have $0 < ba/2D < 1$ so the Lerch function is well defined. Together with another profile, $f(x) = 1$, and some numeric factors, we can recover the result of Ref. [29] following the procedures above.

Back to the exponentially decaying profile, we evaluate the influence of inhomogeneity on the total Casimir force. We focus on the left cavity part. When $a \to \infty$, $\mathrm{Re}[\ln i\beta - \psi(\beta)]$ vanishes, but $[\psi(1) - \ln\frac{\eta}{2n_0} + \frac{11}{6}]$ is nonzero and depends on material's properties n_0, η. This means Eq. (27) contains a part of energy that is not free and thus has no influence to Casimir force. We let $D \to \infty$ to remove the right cavity so that we can focus only on the left cavity (two-plate interaction). The force contributions are

$$F_h = -\frac{A\hbar c\pi^2}{240n_0}\frac{1}{a^4}, \tag{28}$$

$$\delta F_1 = \delta\alpha\frac{A\hbar c\pi^2}{240n_0}\frac{1}{a^4}\left[F_0^{\mathcal{L}} + \frac{1}{3}\left[F_0^{\mathcal{L}} - f(a)\right]\right], \tag{29}$$

$$\delta F_2 = \delta\alpha\frac{A\hbar c\pi^2}{240n_0}\frac{1}{a^4}\left\{\frac{60}{\pi}(e^{-\eta a} - 1)\mathrm{Im}\left[\beta^3[1 - \beta\psi'(\beta)] + \frac{\beta}{6}\right] - \frac{60}{\pi}\eta a e^{-\eta a}\mathrm{Im}\left[\beta^3\left(\psi(1) - \ln\frac{\eta}{2n_0} + \frac{11}{6}\right) + \beta^3[\ln i\beta - \psi(\beta)] - \frac{\beta}{12}\right]\right\}. \tag{30}$$

Equation (28) is the well-known Casimir force between two plates within homogeneous media. If we treat the medium between two plates as homogeneous and use the average refractive index, we can get an approximation of the Casimir force:

$$\bar{F}_h = -\frac{A\hbar c\pi^2}{240n_0}\frac{1}{a^4}\left[1 - \delta\alpha F_0^{\mathcal{L}}\right]. \tag{31}$$

This is exactly the combination of Eq. (28) and the first part of Eq. (29). The rest (second part) of Eq. (29) together with Eq. (30) is written as $\delta\bar{F}$. This term is easy to understand. It reflects the change of the average refractive index when the plate is shifted.

The relation between $\bar{F}_h$ and $\delta\bar{F}$ is given in Fig. 2. We should emphasize that, according to Eqs. (5) and (19), we have $\delta\alpha\sum_\lambda \mathcal{P}_J < 2\delta\alpha$. All $\delta\alpha < 1$ isare permitted within perturbation theory (though the first-order correction might not be enough). Our simulation results clearly show, when $\eta > 10^7\ \mathrm{m}^{-1}$ and $\delta\alpha > 0.2$, the correction even dominates over the homogeneous approximation and flips the sign in some range. When $\delta\alpha = 0.1$, the first-order corrections also have a relative magnitude of peak 50% and thus cannot be ignored. We should take seriously this inhomogeneity-induced repulsion, since this might indicate alternatively the first-order perturbation is too rough for such an intense inhomogeneity. To investigate whether inhomogeneity can induce the repulsive Casimir force and be used to control the Casimir force, we need further studies. On the other hand, when $\delta\alpha \ll 0.01$, or $\eta \ll 5 \times 10^5\ \mathrm{m}^{-1}$, in the range of 0–1 μm where the Casimir force is measurable, the correction is well below 1% and can be omitted. In fact, most experiments under natural conditions belong to this case.

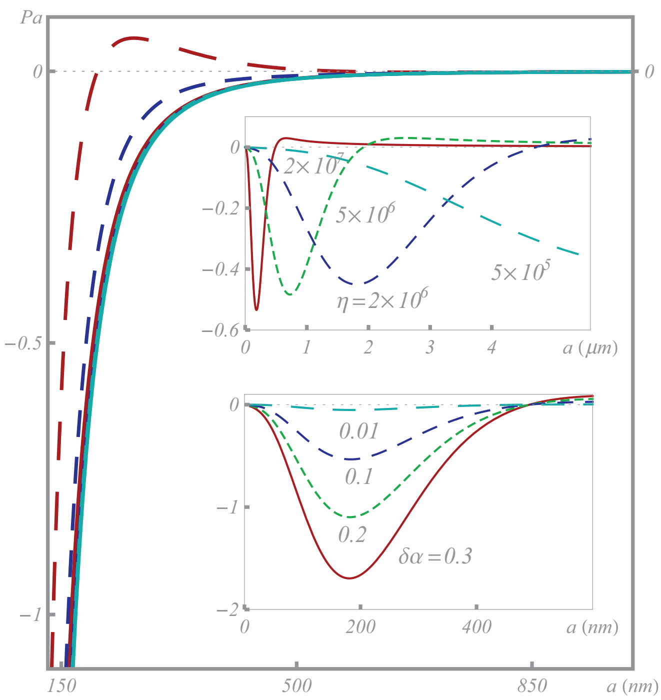

FIG. 2. (Color online) The homogeneous approximation and the exact Casimir pressure including the first-order correction between plates within an inhomogeneous medium. Solid curves are approximations and dashed curves are exact pressures in units of Pascal (minus means attractive). All x axes are the position a. The parameters are $\eta = 10^7\ \mathrm{m}^{-1}$ and $n_0 = 1.5$ for all, while $\delta\alpha = 0.3$, 0.1, and 0.01 for the red (long-dashed), blue (medium-dashed), and cyan (short-dashed) curves, respectively. Red solid, blue solid, cyan solid, and cyan short-dashed curves appear to coincide. Insets are ratios of $\delta\bar{F}/\bar{F}_h$. Top inset: $\delta\alpha = 0.1$, $n_0 = 1.5$, and $\eta = 2\times 10^7$, 5×10^6, 2×10^6, and $5\times 10^5\ \mathrm{m}^{-1}$ for red solid, green short-dashed, blue medium-dashed, and cyan long-dashed curves, respectively. Bottom inset: $\eta = 2\times 10^7\ \mathrm{m}^{-1}$, $n_0 = 1.5$, and $\delta\alpha = 0.3$, 0.2, 0.1, and 0.01 for red solid, green short-dashed, blue medium-dashed, and cyan long-dashed curves, respectively. Zero base lines are plotted as visual guides.

VI. DISCUSSIONS

First, we note the linearly inhomogeneous case $f(x) = x/D$. From Eq. (16) we know $\delta\tilde{E}_2 = 0$; thus there is no logarithm term. This seems to tell us that removing the "bulk contribution" is enough in this case to retrieve finite Casimir force in the half-space model. This actually is not true. A second-order correction, $\langle\chi_J^0|(x/D)^2|\chi_J^0\rangle$, would immediately introduce logarithm terms, let alone higher-order corrections. We thus stress that our results are valid in the first-order approximation.

If the central plate is of finite thickness, the local pressures of the surrounding medium on both sides of the plate do not cancel. Such pressures are worthy of further study. In this paper, we assume the thickness of the plate is infinitesimal and study the interactions between plates to avoid a mixture of problems.

Last, if somehow we knew higher-order corrections would not produce both a- and cutoff-dependent terms, which is a reasonable expectation, the magnitudes of first-order corrections would be quite reliable then.

VII. CONCLUSIONS

With the mode-summation technique and first-order perturbation theory, we have expressed the regularized Casimir energy for the inhomogeneous case with cylinder kernel coefficients, as Eq. (22). Like other unresolved Casimir divergences, we found the presence of the logarithmically cutoff-dependent term [see Eq. (23) and the subsequent analysis]. Our results have also shown there is a term of quadratic cutoff dependence in the Casimir energy. In the piston model such terms are independent of position a and thus vanish in the force, while in the half-space model such terms are dependent on a and thus remain in the force. Consequently, we must introduce additional regularizations to remove them in the half-space model, though it is not clear how to do this.

Based on the piston model, our results have shown, for any smoothly inhomogeneous profile, the Casimir force is always cutoff independent in the first-order perturbation. For some other profiles that are not smooth, it seems one can still get cutoff-independent results, though we cannot give a rigorous proof yet that this is always the case. Our results support the method in Ref. [36] to omit diverging terms when simulating the Casimir force within inhomogeneous media numerically.

We have also calculated the first-order corrections to both the free Casimir energy and the free Casimir force for an exponentially decaying profile. Surprisingly, comparing with the homogeneous analog where the average refractive index between two plates is used, we found the correction to the Casimir force can be even larger than the predicted value of the homogeneous analog and flips the sign of the force, though we note the first-order correction might not be accurate enough. All of these results may be useful as a reference for future relevant theoretical calculations and experimental studies.

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